Algorithms Design and Analysis [ETCS-301]

Dr. A K Yadav
Amity School of Engineering and Technology
(affiliated to GGSIPU, Delhi)
akyadav1@amity.edu
akyadav@akyadav.in
www.akyadav.in
+91 9911375598

October 24, 2019

Iteration method

- In iteration method, expend the right hand side of the recurrence relation.
- Try to find out the series and number of terms in the series.
- Find out the sum of the series

Example 1 for Iteration method I

Example 1 - Find the solution of $T(n) = 2T(n/2) + n$
Take

$$T(n) = 2T(n/2) + n \tag{1}$$

Find the value of $T(n/2) = 2T(n/4) + n/2$ from Eq. 1 and put into Eq. 1

$$\Rightarrow T(n) = 2(2T(n/4) + n/2) + n$$

$$T(n) = 4T(n/4) + 2n \tag{2}$$

Again find the value of $T(n/4) = 2T(n/8) + n/4$ from Eq. 1 and put into Eq. 2

$$\Rightarrow T(n) = 4(2T(n/8) + n/4) + 2n$$

Example 1 for Iteration method II

$$T(n) = 8T(n/8) + 3n \tag{3}$$

Repeating this for i times $T(n)$ will be

$$\Rightarrow T(n) = 2^i T(n/2^i) + in \tag{4}$$

$T(n/2^i)$ will terminate when $n/2^i = 1$ i.e $T(1)$

$\Rightarrow n/2^i = 1$

$\Rightarrow n = 2^i$

$\Rightarrow \lg n = i$

So after $\lg n$ iteration, the recurrence term will be terminated. Eq. 4 will be

$\Rightarrow T(n) = 2^{\lg n} T(1) + n \lg n$

Example 1 for Iteration method III

$\Rightarrow T(n) = nT(1) + n\lg n$

$\Rightarrow T(n) \leq cn + n\lg n, \forall c \geq T(1)$

$\Rightarrow T(n) \leq cn\lg n + n\lg n, \forall n \geq 2$

$\Rightarrow T(n) = (c+1)n\lg n$

$\Rightarrow T(n) \leq c_1 n\lg n, \forall c_1 \geq (c+1) > 0, \forall n \geq 2$

$\therefore T(n) = O(n\lg n)$

Example 2 for Iteration method I

Example 2 - Find the solution of $T(n) = 3T(n/4) + \Theta(n^2)$
Take for $c > 0$

$$T(n) = 3T(n/4) + cn^2 \tag{5}$$

Find the value of $T(n/4) = 3T(n/16) + c(n/4)^2$ from Eq. 5 and put into Eq. 5

$$\Rightarrow T(n) = 3\left(3T(n/16) + c(n/4)^2\right) + cn^2$$

$$T(n) = 9T(n/16) + \left(1 + \frac{3}{16}\right) cn^2 \tag{6}$$

Again find the value of $T(n/16) = 3T(n/64) + c(n/16)^2$ from Eq. 5 and put into Eq. 6

$$\Rightarrow T(n) = 9\left(3T(n/64) + c(n/16)^2\right) + \left(1 + \frac{3}{16}\right)cn^2$$

$$T(n) = 27T(n/64) + \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2\right)cn^2 \tag{7}$$

Repeating this for i times $T(n)$ will be

$$\Rightarrow T(n) = 3^i T(n/4^i) + \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots + \left(\frac{3}{16}\right)^{(i-1)}\right)cn^2 \tag{8}$$

$T(n/4^i)$ will terminate when $n/4^i = 1$ i.e $T(1)$

$\Rightarrow n/4^i = 1$

$\Rightarrow n = 4^i$

$\Rightarrow \log_4 n = i$

So after $\log_4 n$ iteration, the recurrence term will be terminated. Eq. 8 will be

$$\Rightarrow T(n) = 3^{\log_4 n} T(1) + \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots + \left(\frac{3}{16}\right)^{(\log_4 n - 1)}\right) cn^2$$

$$\Rightarrow T(n) = n^{\log_4 3} T(1) + \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots + \left(\frac{3}{16}\right)^{(\log_4 n - 1)}\right) cn^2$$

Example 2 for Iteration method IV

$$\Rightarrow T(n) \leq c_1 n + \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \dots\right) cn^2, \forall c_1 \geq T(1)$$

$$\Rightarrow T(n) \leq c_1 n + \left(\frac{1}{1 - 3/16}\right) cn^2$$

$$\Rightarrow T(n) \leq c_1 n + \left(\frac{16}{13}\right) cn^2$$

$$\Rightarrow T(n) \leq c_1 n^2 + \left(\frac{16}{13}\right) cn^2, \ \forall n \geq 1$$

$$\Rightarrow T(n) \leq \left(\frac{13c_1 + 16c}{13}\right) n^2$$

Example 2 for Iteration method V

$$\Rightarrow T(n) \leq c_2 n^2 \text{ where } \forall c_2 \geq \frac{13c_1 + 16c}{13} > 0 \text{ and } \forall n \geq 1$$

$$\therefore T(n) = O(n^2)$$

Example 3 - Find the solution of $T(n) = T(n-1) + n^4$

$$T(n) = T(n-1) + n^4 \tag{9}$$

Find the value of $T(n-1) = T(n-2) + (n-1)^4$ from Eq. 9 and put into Eq. 9

$$\Rightarrow T(n) = \left(T(n-2) + (n-1)^4\right) + n^4$$

$$T(n) = T(n-2) + (n-1)^4 + n^4 \tag{10}$$

Again find the value of $T(n-2) = T(n-3) + (n-2)^4$ from Eq. 9 and put into Eq. 10

$$\Rightarrow T(n) = \left(T(n-3) + (n-2)^4\right) + (n-1)^4 + n^4$$

$$T(n) = T(n-3) + (n-2)^4 + (n-1)^4 + n^4 \tag{11}$$

Repeating this for i times $T(n)$ will be

$$\Rightarrow T(n) = T(n-i) + (n-i+1)^4 + \cdots + (n-1)^4 + n^4 \tag{12}$$

$T(n-i)$ will terminate when $n-i=0$ i.e $T(0)$

$$\Rightarrow n-i=0$$

$$\Rightarrow i=n$$

So after n iteration, the recurrence term will be terminated. Eq. 12 will be

$$\Rightarrow T(n) = T(0) + 1^4 + 2^4 + 3^4 + \cdots + (n-1)^4 + n^4$$

Example 3 for Iteration method III

$$\Rightarrow T(n) \leq T(0) + n^4 + n^4 + \cdots + n^4, \forall n \geq 1$$

$$\Rightarrow T(n) \leq T(0) + nn^4$$

$$\Rightarrow T(n) \leq cn^5 + n^5, \forall c \geq T(0)$$

$$\Rightarrow T(n) \leq (c+1)n^5$$

$$\Rightarrow T(n) \leq c_1 n^5 \text{ where } \forall c_1 \geq (c+1) > 0 \text{ and } \forall n \geq 1$$

$$\therefore T(n) = O(n^5)$$

Example 4 - Find the solution of $T(n) = T(\sqrt{n}) + n$

$$T(n) = T(\sqrt{n}) + n \tag{13}$$

Find the value of $T(\sqrt[2]{n}) = T(\sqrt[4]{n}) + \sqrt[2]{n}$ from Eq. 13 and put into Eq. 13

$$\Rightarrow T(n) = (T(\sqrt[4]{n}) + \sqrt[2]{n}) + n$$

$$T(n) = T(\sqrt[4]{n}) + \sqrt[2]{n} + n \tag{14}$$

Again find the value of $T(\sqrt[4]{n}) = T(\sqrt[8]{n}) + \sqrt[4]{n}$ from Eq. 13 and put into Eq. 14

$$\Rightarrow T(n) = (T(\sqrt[8]{n}) + \sqrt[4]{n}) + \sqrt[2]{n} + n$$

Example 4 for Iteration method II

$$T(n) = T(\sqrt[8]{n}) + \sqrt[4]{n} + \sqrt[2]{n} + n \tag{15}$$

Repeating this for i times $T(n)$ will be

$$\Rightarrow T(n) = T(\sqrt[2^i]{n}) + \sqrt[2^{i-1}]{n} + \cdots + \sqrt[4]{n} + \sqrt[2]{n} + n \tag{16}$$

$T(\sqrt[2^i]{n})$ will terminate when $\sqrt[2^i]{n} = 2$ i.e $T(2)$

$$\Rightarrow \sqrt[2^i]{n} = 2$$

$$\Rightarrow \lg(\sqrt[2^i]{n}) = \lg 2$$

$$\Rightarrow \frac{1}{2^i} \lg n = 1$$

$$\Rightarrow \lg n = 2^i$$

$$\Rightarrow \lg \lg n = i$$

So after $\lg \lg n$ iteration, the recurrence term will be terminated. Eq. 16 will be

$$\Rightarrow T(n) = n + \sqrt[2]{n} + \sqrt[4]{n} + \cdots + \sqrt[2^{i-1}]{n} + T(2)$$

$$\Rightarrow T(n) \leq n + n + n + \ldots \quad \lg \lg n \text{ times}$$

$$\Rightarrow T(n) \leq n \lg \lg n$$

$$\therefore T(n) = O(n \lg \lg n)$$

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598