Algorithms Design and Analysis [ETCS-301]

Dr. A K Yadav
Amity School of Engineering and Technology
(affiliated to GGSIPU, Delhi)
akyadav1@amity.edu
akyadav@akyadav.in
www.akyadav.in
+91 9911375598

October 23, 2019

Bellman-Ford algorithm

- A graph with $|V|$ vertices can have maximum $|V|-1$ edges and $|V|$ vertices in any simple path.
- If number of edges in the path is greater than $|V|-1$ or vertices on the path is greater than $|V|$ including source and destination then there is cycle in path.
- Bellman works upon this fundamental and relaxes all edges $|V|-1$ times to find the shortest path from single source.
- After $|V|-1$ passes, there will be shortest path among nodes form source s, if exists.
- One more cycle of relax, if decrease the shortest path from source then it means there will be $|V|$ edges and $|V|+1$ vertices and hence a negative weight cycle.
- The algorithm returns either TRUE if there is no negative weight cycle or FALSE if negative weight cycle

Algorithm

BELLMAN-FORD(G,w,s)

1. INITIALIZE-SINGLE-SOURCE(G, s)
2. for $i = 1$ to $|V| - 1$
3.   for each edge $(u, v) \in E$
4.   RELAX(u, v, w)
5. for each edge $(u, v) \in E$
6.   if $d[v] > d[u] + w(u, v)$
7.   return FALSE
8. return TRUE

Time complexity of the algorithm is $O(VE)$

Correctness of the Bellman-Ford algorithm I

Statement: If G contains no negative-weight cycles that are reachable from s, then the algorithm returns TRUE, we have $d[v] = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s. If G does contain a negative-weight cycle reachable from s, then the algorithm returns FALSE.

- Suppose that graph G contains no negative-weight cycles that are reachable from the source s.
- Now we have to prove that the algorithms will return TRUE.
- If vertex v is reachable from s, then at termination after $|V| - 1$ iterations $d[v] = \delta(s, v)$
- If vertex v is unreachable from s, then at termination after $|V| - 1$ iterations $d[v] = \delta(s, v) = \infty$

Correctness of the Bellman-Ford algorithm II

- After termination of $|V|-1$ iterations, for all edge $(u,v) \in E, d[v] = \delta(s,v)$

  $d[v] = \delta(s,v)$

  $d[u] = \delta(s,u)$

  $d[v] = \delta(s,v) \leq \delta(s,u) + w(u,v)$-by the triangle inequality

  $d[v] \leq d[u] + w(u,v)$
- So $d[v] \leq d[u] + w(u,v)$ for all edge $(u,v) \in E$ in the predecessor subgraph G_π is a shortest path from source s
- So $d[v] > d[u] + w(u,v)$ of line no **??** of the algorithm will never TRUE and hence algorithm will return TRUE.
- Suppose that graph G contains negative-weight cycles that are reachable from the source s.

Correctness of the Bellman-Ford algorithm III

- Now we have to prove that the algorithms will return FALSE.
- Let $c = \langle v_0, v_1, \ldots, v_k \rangle$ is a negative weight cycle in the path p reachable from source s in graph G where $v_0 = v_k$
- $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0 \leftarrow$ because of negative weight cycle
- Now let contradiction that there is a negative weight cycle but still algorithm returns TRUE.
- So $d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$.
- Taking summation over all the k values

$$\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

Correctness of the Bellman-Ford algorithm IV

- $d[v_1] + d[v_2] + \cdots + d[v_{k-1}] + d[v_k]$

  $$\leq d[v_0] + d[v_1] + \cdots + d[v_{k-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- $0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i), \because v_0 = v_k$
- But $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$ because of negative weight cycle
- So our assumption of contradiction is wrong and the algorithm will return FALSE.

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598