Algorithms Design and Analysis [ETCS-301]

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Let $a > 0$ and $b > 0$ be constants, let $f(n)$ be an asymptotically positive function such that $f(n) = O(n^d)$ for $d \geq 0$ and let $T(n)$ be defined on the positive integers by the recurrence

$$T(n) = aT(n-b) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

1. if $a < 1$ then $T(n) = O(n^d)$
2. if $a = 1$ then $T(n) = O(n^{d+1})$
3. if if $a > 1$ then $T(n) = O(a^{\frac{n}{b}} n^d)$

Muster Theorem (S&C) II

Proof:
Expend the left hand side term of given recurrence

$$T(n) = aT(n-b) + f(n)$$

$$\Rightarrow T(n) = a^2T(n-2b) + af(n-b) + f(n)$$

$$\Rightarrow T(n) = a^3T(n-3b) + a^2f(n-2b) + af(n-b) + f(n)$$

$$\Rightarrow T(n) = a^iT(n-ib) + a^{i-1}f(n-(i-1)b) + \cdots + af(n-b) + f(n)$$

This will be terminated when $n - ib = 0$ that is $i = \frac{n}{b}$

$$\Rightarrow T(n) = a^{\frac{n}{b}}T(0) + a^{\frac{n}{b}-1}f\left(n - \left(\frac{n}{b} - 1\right)b\right) + \cdots + af(n-b) + f(n)$$

$$\Rightarrow T(n) = \sum_{i=0}^{\frac{n}{b}-1} a^if(n-ib) + a^{\frac{n}{b}}T(0)$$

$$\Rightarrow T(n) \leq \sum_{i=0}^{\frac{n}{b}} a^i f(n - ib)$$

$$\Rightarrow T(n) = O\left(n^d \sum_{i=0}^{\frac{n}{b}} a^i\right)$$

Now there are 3 cases:

1. if $a < 1$ then

$$\sum_{i=0}^{\frac{n}{b}} a^i = \frac{1 - a^{\frac{n}{b}+1}}{1 - a} \leq \frac{1}{1 - a} = O(1)$$

$$\Rightarrow T(n) = O\left(n^d\right) O(1)$$

$$\Rightarrow T(n) = O\left(n^d\right)$$

2. if $a = 1$ then

$$\sum_{i=0}^{\frac{n}{b}} a^i = \sum_{i=0}^{\frac{n}{b}} 1 = \frac{n}{b} + 1 = O(n)$$

$$\Rightarrow T(n) = O\left(n^d\right) O(n)$$

$$\Rightarrow T(n) = O\left(n^{d+1}\right)$$

3. if $a > 1$ then

$$\sum_{i=0}^{\frac{n}{b}} a^i = \frac{a^{\frac{n}{b}+1} - 1}{a - 1} = O(a^{\frac{n}{b}})$$

$$\Rightarrow T(n) = O\left(n^d\right) O(a^{\frac{n}{b}})$$

$$\Rightarrow T(n) = O\left(n^d a^{\frac{n}{b}}\right)$$

so

$$T(n) = \begin{cases} O(n^d) & a < 1 \\ O(n^{d+1}) & a = 1 \\ O(a^{\frac{n}{b}} n^d) & a > 1 \end{cases}$$

Examples of Muster Theorem (S&C)

1. Find the solution of $T(n) = \frac{1}{2}T(n-1) + n$
2. Find the solution of $T(n) = \frac{7}{8}T(n-1) + n^3$
3. Find the solution of $T(n) = T(n-2) + n$
4. Find the solution of $T(n) = T(n-2) + n \lg n$
5. Find the solution of $T(n) = T(n-2) + \frac{n}{\lg n}$
6. Find the solution of $T(n) = 2T(n-2) + n$
7. Find the solution of $T(n) = 3T(n-1) + n^2$

Solutions of Examples of Muster Theorem (S&C) I

1. Find the solution of $T(n) = \frac{1}{2}T(n-1) + n$
   Here $a = \frac{1}{2}, b = 1, f(n) = O(n), d = 1,$
   Case 1 applies. So the solution is :

$$T(n) = O\left(n^d\right)$$

$$T(n) = O(n)$$

2. Find the solution of $T(n) = \frac{7}{8}T(n-1) + n^3$
   Here $a = \frac{7}{8}, b = 1, f(n) = O(n^3), d = 3,$
   Case 1 applies. So the solution is :

$$T(n) = O\left(n^d\right)$$

$$T(n) = O\left(n^3\right)$$

3. Find the solution of $T(n) = T(n-2) + n$
   Here $a = 1, b = 2, f(n) = O(n), d = 1,$
   Case 2 applies. So the solution is :

   $$T(n) = O\left(n^{d+1}\right)$$

   $$T(n) = O\left(n^{2}\right)$$

4. Find the solution of $T(n) = T(n-2) + n \lg n$
   Here $f(n) = n \lg n$, function is not polynomial of n so method cann't be applied.
   $a = 1, b = 2, f(n) = O(n^2), d = 2,$
   Case 2 applies. So the solution is :

   $$T(n) = O\left(n^{d+1}\right)$$

   $$T(n) = O\left(n^{3}\right)$$

5. Find the solution of $T(n) = T(n-2) + \frac{n}{\lg n}$
   Here $f(n) = \frac{n}{\lg n}$, function is not polynomial of n so method cann't be applied.
   $a = 1, b = 2, f(n) = O(n), d = 1,$
   Case 2 applies. So the solution is :

   $$T(n) = O\left(n^{d+1}\right)$$

   $$T(n) = O\left(n^{2}\right)$$

6. Find the solution of $T(n) = 2T(n-2) + n$
   Here $a = 2, b = 2, f(n) = O(n), d = 1,$
   Case 3 applies. So the solution is :

   $$T(n) = O\left(n^{d} a^{\frac{n}{b}}\right)$$

   $$T(n) = O\left(n^{1} 2^{\frac{n}{2}}\right)$$

7. Find the solution of $T(n) = 3T(n-1) + n^2$
   Here $a = 3, b = 1, f(n) = O(n^2), d = 2,$
   Case 3 applies. So the solution is :

$$T(n) = O\left(n^d a^{\frac{n}{b}}\right)$$

$$T(n) = O\left(n^2 3^n\right)$$

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598