Undecidability

Module 4

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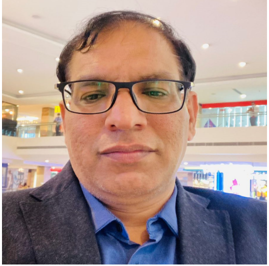

April 28, 2022

Outline

1. Turing machine halting Problem
2. Post correspondence problems (PCP)
3. Modified Post correspondence problems

Some basic definition

- When a Turing machine reaches a final state, it halts.
- We can also say that a Turing machine M halts when M reaches a state q and a current symbol a to be scanned so that $\delta(q, a)$ is undefined.
- There are TMs that never halt on some inputs in any one of these ways.
- So we make a distinction between the languages accepted by a TM that halts on all input strings and a TM that never halts on some input strings.
- Recursively Enumerable: A language $L \subseteq \Sigma^*$ is recursively enumerable if there exists a TM M, such that $L = T(M)$.
- Recursive: A language $L \subseteq \Sigma^*$ is recursive if there exists some TM M that satisfies the following two conditions:
  1. If $w \in L$ then M accepts w, that is. reaches an accepting state on processing w and halts.

Some basic definition

2. If $w \notin L$ then M eventually halts, without reaching an accepting state.

- Decidable:A problem with two answers (Yes/No) is decidable if the corresponding language is recursive. In this case, the language L is also called decidable.
- Undecidable:A problem/language is undecidable if it is not decidable.
- A decidable problem is called a solvable problem and an undecidable problem an unsolvable problem by some authors.
- $A_{DFA} = \{(B, w) | B$ accepts the input string $w\}$
- $A_{CFG} = \{(G, w) |$ The context-free grammar G accepts the input string $w\}$
- $A_{CSG} = \{(G, w) |$ The context-sensitive grammar G accepts the input string $w\}$

Some basic definition

- $A_{TM} = \{(M, w)|$ The TM M accepts $w\}$
- A_{DFA} is decidable.
- A_{CFG} is decidable.
- A_{CSG} is decidable.
- A_{TM} is undecidable.

Turing machine halting Problem

- The reduction technique is used to prove the undecidability of halting problem of Turing machine
- We say that problem A is reducible to problem B if a solution to problem B can be used to solve problem A.
- If A is reducible to B and B is decidable then A is decidable. If A is reducible to B and A is undecidable, then B is undecidable.
- Theorem $HALT_{TM} = \{(M, w)|$ The Turing machine M halts on input $w\}$ is undecidable.
  Proof: We assume that $HALT_{TM}$ is decidable, and get a contradiction. Let M_1 be the TM such that $T(M_1) = HALT_{TM}$ and let M_1 halt eventually on all (M, w). We construct a TM M_2 as follows:
  1. For $M_2, (M, w)$ is an input.
  2. The TM M_1 acts on (M, w).

Turing machine halting Problem

3. If M_1 rejects (M, w) then M_2 rejects (M, w).
4. If M_1 accepts (M, w), simulate the TM M on the input string w until M halts.
5. If M has accepted w, M_2 accepts (M, w); otherwise M_2 rejects (M, w).

- When M_1 accepts (M, w) (in step 4), the Turing machine M halts on w.
- In this case either an accepting state q or a state q' such that $\delta(q', a)$ is undefined till some symbol a in w is reached.
- In the first case (the first alternative of step 5) M_2 accepts $(M.w)$.
- In the second case (the second alternative of step 5) M_2 rejects (M, w).
- It follows from the definition of M_2 that M_2 halts eventually.
- $TM_2 = \{(M, w) |$ The Turing machine accepts $w\} = A_{TM}$
- This is a contradiction since A_{TM} is undecidable.

Post correspondence problems (PCP)

- The Post Correspondence Problem (PCP) was first introduced by Emil Post in 1946.
- The problem over an alphabet Σ belongs to a class of yes/no problems and is stated as follows:
- Consider the two lists $x = (x_1 \dots x_n), y = (y_1 \dots y_n)$ of non-empty strings over an alphabet $\Sigma = \{0, 1\}$.
- The PCP is to determine whether or not there exist $i_1, \dots, i_m$, where $1 \leq i_j \leq n$ such that

$$x_{i_1} \dots x_{i_m} = y_{i_1} \dots y_{i_m}$$

- The indices i_j's need not be distinct and m may be greater than n. Also, if there exists a solution to PCP, there exist infinitely many solutions.

Modified Post correspondence problems

- If the first substring used in PCP is always x_1 and y_1 then the PCP is known as the Modified Post Correspondence Problem.

Questions?

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Thank you.