Algorithms Design and Analysis [ETCS-301]

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Shortest-paths problem

- G is a given weighted and directed graph with set of vertices V and set of edges E such that $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$
- The weight $w(p)$ of path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of edges of the path. That is

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- The shortest-path weight $\delta(u, v)$ from u to v is defined as

$$\delta(u, v) = \begin{cases} min\{w(p) : u \rightsquigarrow v & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- A shortest path from vertex u to vertex v is any path p with weight $w(p) = \delta(u.v)$

Variants of Shortest path

- Single-source shortest-paths problem : Find a shortest path from a given source vertex $s \in V$ to each vertex $v \in V$
- Single-destination shortest-paths problem: Find a shortest path from each vertex $u \in V$ to a given destination vertex $d \in V$
- Single-pair shortest-path problem: Find a shortest path from a given source $s \in V$ to a given destination $d \in V$.
- All-pairs shortest-paths problem: Find a shortest path from each vertex $u \in V$ to each vertex $v \in V$.

Optimal substructure of a shortest path I

Statement: If p is a optimal path from v_0 to v_k then p_{ij} will be the optimal path from p_i to p_j for any value of i and j such that $0 \leq i \leq j \leq k$.

Proof:

- Given p is the optimal path from v_0 to v_k that is $w(p)$ is minimum over $v_0 \rightsquigarrow v_k$
- Let's break path p into 3 parts :
- p_{0i} is the path from v_0 to v_i with weight $w(p_{0i})$
- p_{ij} is the path from v_i to v_j with weight $w(p_{ij})$
- p_{jk} is the path from v_j to v_k with weight $w(p_{jk})$
- That is $v_0 \rightsquigarrow v_i \rightsquigarrow v_j \rightsquigarrow v_k$
- Now $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$
- Let $w(p'_{ij})$ is the minimum weight path from v_i to v_j through other vertices that is $w(p'_{ij}) \leq w(p_{ij})$

Optimal substructure of a shortest path II

- So $w(p^{'}) = w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$
- As $w(p'_{ij}) \leq w(p_{ij})$ so $w(p^{'}) \leq w(p)$
- But $w(p)$ is the optimal path so $w(p^{'}) \not< w(p)$
- The only possible case is $w(p^{'}) = w(p)$
- It means $w(p'_{ij}) = w(p_{ij})$

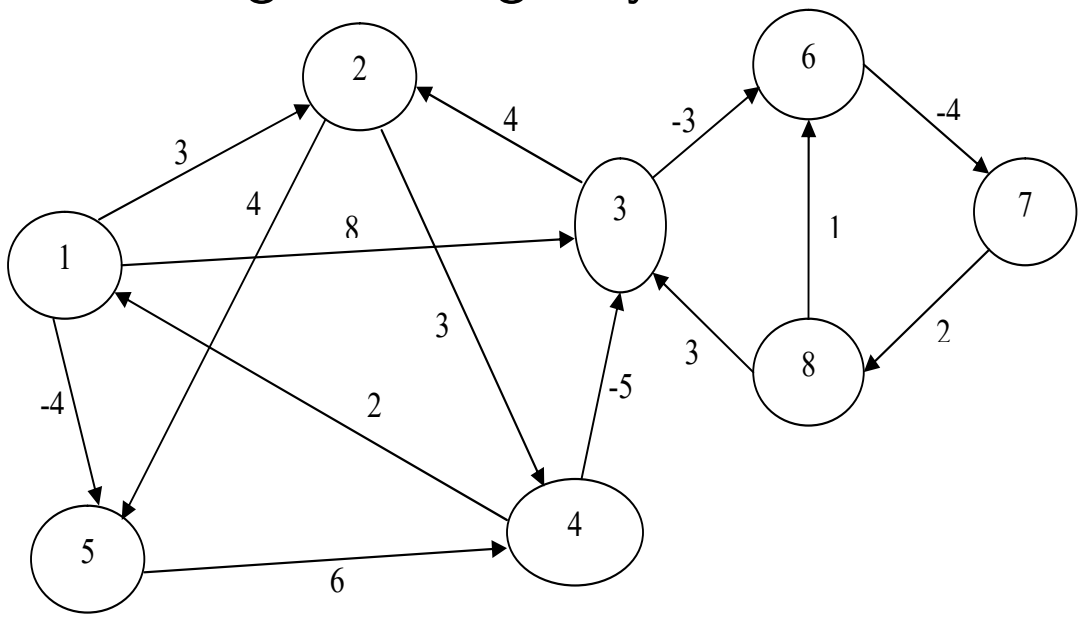
So Subpaths of shortest paths are shortest paths.

Negative-weight cycle I

- An edge with with negative weight is called negative-weight edge.
- If $w(p)$ is less than 0 ($w(p) < 0$) then it is known as negative-weight cycle
- If no negative-weight cycles reachable from the source s then shortest path said to well defined
- If negative-weight cycles reachable from the source s then shortest path said to not well defined
- If shortest path is well defined then $\delta(u, v) =$"some finite value"
- If shortest path is not well defined then $\delta(u, v) = -\infty$
- It means no path from s to a vertex on the cycle can be a shortest path

Negative-weight cycle II

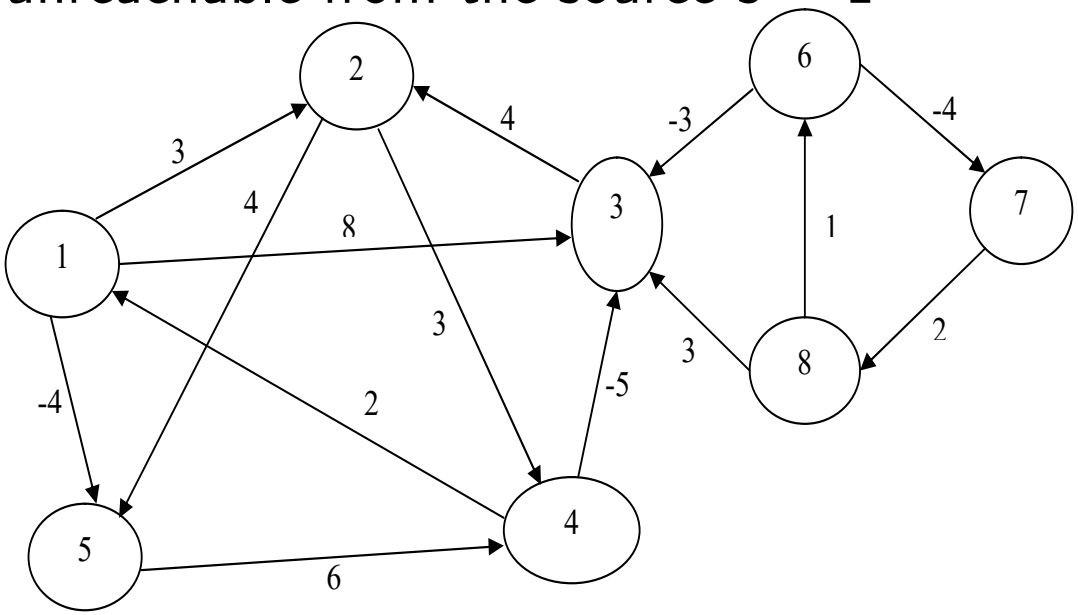
- That is we can always find a path with lower weight by just traversing the negative-weight cycle once more.
- See below figure for negative-weight cycle reachable from the source $s = 1$



- But if negative-weight cycles unreachable from the source $s = 1$ then also we can find the shortest path

Negative-weight cycle III

- Please see below figure for negative-weight cycle cycles unreachable from the source $s = 1$



- There is no path from source $s = 1$ to the unreachable nodes. so $w(p) = \infty$ in this case
- Any acyclic path can have maximum $|V| - 1$ edges and $|V|$ vertices.

Representing shortest paths I

- G is a graph with $G = (V, E)$. $\pi[v]$ is a predecessor of node v. It means there is edge between $(\pi[v], v)$
- **Predecessor subgraph** $G_\pi = (V_\pi, E_\pi)$ is subgraph of G taking s as source where:.
- The vertex set V_π to be the set of vertices of G with non-NIL predecessors, plus the source s:
  $V_\pi = \{v \in V : \pi[v] \neq NIL\} \cup \{s\}$
- The directed edge set E_π is the set of edges induced by the π values for vertices in V_π:
  $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$

Representing shortest paths II

- **Shortest-paths tree** is a directed subgraph $G' = (V', E')$ with s as root where $V' \subseteq V$ and $E' \subseteq E$ such that :
  1. G' forms a rooted tree with root s
  2. V' is the set of vertices reachable from s in G
  3. $\forall v \in V'$, the unique simple path from s to v in G' is a shortest path from s to v in G
- Shortest paths are not necessarily unique and neither are shortest-paths trees.
- $d[v]$ is the shortest-path estimate between the source s and each vertex $v \in V$
- We initialize the shortest-path estimates and predecessors as : **INITIALIZE-SINGLE-SOURCE(G,s)**
  1. for each vertex $v \in V$
  2. $\quad d[v] = \infty$
  3. $\quad \pi[v] = NIL$

Representing shortest paths III

4. $d[s] = 0$

- **Relaxation**: Relaxing an edge (u, v) means testing whether we can improve the shortest path to v found so far by going through u, if yes then update $d[v]$ and $\pi[v]$
- The following code performs a relaxation step on edge (u, v) **RELAX(u,v,w)**
  1. if $d[v] > d[u] + w(u, v)$
  2. $\quad d[v] = d[u] + w(u, v)$
  3. $\quad \pi[v] = u$

Properties of shortest paths I

- **Triangle inequality:** For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$
- **Upper-bound property:** $\forall v \in V, d[v] \geq \delta(s, v)$ and once $d[v] = \delta(s, v)$ then it never changes
- **No-path property:** If there is no path from s to v, then we always have $d[v] = \delta(s, v) = \infty$
- **Convergence property:** If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$ and if $d[u] = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $d[v] = \delta(s, v)$ at all times afterward.
- **Path-relaxation property:** If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $d[v_k] = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Properties of shortest paths II

- **Predecessor-subgraph property:** Once $f[v] = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598