Algorithms Design and Analysis [ETCS-301]

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Recurrences

A recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs.

$$f(n) = \begin{cases} nf(n-1) & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

$$T(n) = \begin{cases} 2T(n/2) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1 \end{cases}$$

$$T(n) = \begin{cases} T(n-1) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1 \end{cases}$$

How to solve recurrence relations?

- Substitution method
- Iteration method
- Recursion tree method
- Master method (D&C)
- Master method (S&C)

Substitution method

- In Substitution method there are two steps :
1 - First guess the solution
2 - Second use mathematical induction to find the constants and show that the solution works.
- This method is powerful, but depends on the correctness of the guess.
- Method can be used for either upper or lower bounds on a recurrence.

Example of Substitution method

Example 1. Prove that the solution of $T(n) = 2T(n/2) + n$ is $O(n \lg n)$

Example 2. Find the solution of the recurrence relation $T(n) = 2T(\sqrt{n}) + \lg n$

Example 3. Show that the solution of $T(n) = 2T(n/2) + n$ is $\Omega(n \lg n)$

Example 4. Show that the solution of $T(n) = T(n/2) + 1$ is $O(\lg n)$

Example 5. Find the solution of $T(n) = T(n-1) + n$.

Solution Example 1 of Substitution method I

Example 1. Prove that the solution of $T(n) = 2T(n/2) + n$ is $O(n \lg n)$

Solution:

Soln is already given so no need to guess the soln. We have to prove that $T(n) \leq cn \lg n$ for some constant $c > 0$ and for all $n > n_0$.

$$T(n) = 2T(n/2) + n \tag{1}$$

$$T(n) = O(n \lg n) \Rightarrow T(n) \leq cn \lg n \tag{2}$$

$\Rightarrow T(n) = O(n \lg n)$ then it must be the solution for the input size $n/2$.

substitute the value of n as $n/2$ in Eq. 2.

Solution Example 1 of Substitution method II

$$T(n/2) \leq c(n/2)\lg(n/2) \tag{3}$$

Substitute the value of $T(n/2)$ from Eq. 3 into Eq. 1.

$$T(n) \leq 2(c(n/2)\lg(n/2)) + n$$

$$\Rightarrow T(n) \leq cn\lg(n/2) + n$$

$$\Rightarrow T(n) \leq cn\lg n - cn\lg 2 + n$$

$$\Rightarrow T(n) \leq cn\lg n - cn + n$$

$$\Rightarrow T(n) \leq cn\lg n - n(c-1)$$

$$\Rightarrow T(n) \leq cn\lg n \ \forall c \geq 1, n \geq 2$$

$$\therefore T(n) = O(n\lg n)$$

Solution Example 2 of Substitution method I

Example 2. Find the solution of the recurrence relation
$T(n) = 2T(\sqrt{n}) + \lg n$
Solution:
Step 1 is to guess the solution but it is very hard to guess for the given recurrence relation.
So First we have to simplify the relation by changing variables.

$$T(n) = 2T(\sqrt{n}) + \lg n \text{ (given)} \tag{4}$$

Take:

$$\lg n = m \tag{5}$$

$$\Rightarrow n = 2^m \tag{6}$$

Solution Example 2 of Substitution method II

Put value of n from Eq. 6 into Eq. 4

$$T(2^m) = 2T(2^{m/2}) + m \tag{7}$$

Rename the recurrence relation Eq. 7 as $S(m) = T(2^m)$ and $S(m/2) = T(2^{m/2})$

$$\Rightarrow S(m) = 2S(m/2) + m \tag{8}$$

Now we can guess the solution as

$$S(m) = O(m \lg m) \tag{9}$$

Put back the value of $S(m) = T(2^m)$ and m from Eq. 5 into Eq. 9

$$T(n) = O(\lg n \lg \lg n) \tag{10}$$

Solution Example 3 of Substitution method I

Example 3. Show that the solution of $T(n) = 2T(n/2) + n$ is $\Omega(n \lg n)$
Solution:
We have to prove that $T(n) \geq cn \lg n$ for some constant $c > 0$ and for all $n > n_0$.

$$T(n) = 2T(n/2) + n \tag{11}$$

$$T(n) = \Omega(n \lg n) \Rightarrow T(n) \geq cn \lg n \tag{12}$$

$\Rightarrow T(n) = \Omega(n \lg n)$ then it must be the solution for the input size $n/2$.
substitute the value of n as $n/2$ in Eq. 12.

Solution Example 3 of Substitution method II

$$T(n/2) \geq c(n/2)\lg(n/2) \tag{13}$$

Substitute the value of $T(n/2)$ from Eq. 13 into Eq. 11.

$$T(n) \geq 2(c(n/2)\lg(n/2)) + n$$

$$\Rightarrow T(n) \geq cn\lg(n/2) + n$$

$$\Rightarrow T(n) \geq cn\lg n - cn\lg 2 + n$$

$$\Rightarrow T(n) \geq cn\lg n - cn + n$$

$$\Rightarrow T(n) \geq cn\lg n + n(1-c)$$

$$\Rightarrow T(n) \geq cn\lg n \ \forall c \leq 1, n \geq 2$$

$$\therefore T(n) = \Omega(n\lg n)$$

Solution Example 4 of Substitution method I

Example 4. Show that the solution of $T(n) = T(n/2) + 1$ is $O(\lg n)$
Solution:
We have to prove that $T(n) \leq c \lg n$ for some constant $c > 0$ and for all $n > n_0$.

$$T(n) = T(n/2) + 1 \tag{14}$$

$$T(n) = O(\lg n) \Rightarrow T(n) \leq c \lg n \tag{15}$$

$\Rightarrow T(n) = O(\lg n)$ then it must be the solution for the input size $n/2$.
substitute the value of n as $n/2$ in Eq. 15.

Solution Example 4 of Substitution method II

$$T(n/2) \leq c \lg(n/2) \tag{16}$$

Substitute the value of $T(n/2)$ from Eq. 16 into Eq. 14.

$$T(n) \leq c \lg(n/2)) + 1$$

$$\Rightarrow T(n) \leq c \lg(n/2) + 1$$

$$\Rightarrow T(n) \leq c \lg n - c \lg 2 + 1$$

$$\Rightarrow T(n) \leq c \lg n - c + 1$$

$$\Rightarrow T(n) \leq c \lg n - (c - 1)$$

$$\Rightarrow T(n) \leq c \lg n \; \forall c \geq 1, n \geq 2$$

$$\therefore T(n) = O(\lg n)$$

Solution Example 5 of Substitution method I

Example 5. Find the solution of $T(n) = T(n-1) + n$.
Solution:
Guess the solution be $O(n^2)$. Now we have to prove that $T(n) \leq cn^2$ for some constant $c > 0$ and for all $n > n_0$.

$$T(n) = T(n-1) + n \tag{17}$$

$$T(n) = O(n^2) \Rightarrow T(n) \leq cn^2 \tag{18}$$

$\Rightarrow T(n) = O(n^2)$ then it must be the solution for the input size $n-1$.
substitute the value of n as $n-1$ in Eq. 18.

$$T(n-1) \leq c(n-1)^2 \tag{19}$$

Solution Example 5 of Substitution method II

Substitute the value of $T(n-1)$ from Eq. 19 into Eq. 17.

$$T(n) \leq c(n-1)^2 + n$$

$$\Rightarrow T(n) \leq c(n^2 - 2n + 1) + n$$

$$\Rightarrow T(n) \leq cn^2 - 2cn + c + n$$

$$\Rightarrow T(n) \leq cn^2 - (2c-1)n + c$$

Take value of c such that $(2c-1) \geq c$

$$\Rightarrow T(n) \leq cn^2 \ \forall c \geq 1, n \geq 1$$

$$\therefore T(n) = O(n^2)$$

ज्योतिर्वृणीत तमसो विजानन

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598