Algorithms Design and Analysis [ETCS-301]

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Master method is based on the Master Theorem, which states as:
Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be an asymptotically positive function, and let $T(n)$ be defined on the non-negative integers by the recurrence

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Then $T(n)$ has the following asymptotic bounds:

1. if $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$
3. if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Examples of Master method (D&C)

1. Find the solution of $T(n) = 9T\left(\frac{n}{3}\right) + n$
2. Find the solution of $T(n) = T\left(\frac{2n}{3}\right) + 1$
3. Find the solution of $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$
4. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + n\lg n$
5. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$
6. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$
7. Find the solution of $T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$
8. Find the solution of $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^3)$

1. Find the solution of $T(n) = 9T\left(\frac{n}{3}\right) + n$

Comparing $T(n) = 9T\left(\frac{n}{3}\right) + n$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 9 \geq 1, b = 3 > 1$ and $f(n) = n$ so we can apply Master Method to solve this recursion.
As $n^{\log_b a} = n^{\log_3 9} = n^2$ is polynomially greater then $f(n) = n$.
so $n = O\left(n^{\log_3 9 - \epsilon}\right)$ where for any $0 < \epsilon \leq 1$.
Case 1 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(n^{\log_3 9}\right) = \Theta\left(n^2\right)$$

Solution of Examples 2 of Master method (D&C)

2. Find the solution of $T(n) = T\left(\frac{2n}{3}\right) + 1$

Comparing $T(n) = T\left(\frac{2n}{3}\right) + n$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 1 \geq 1, b = \frac{3}{2} > 1$ and $f(n) = n^0 = 1$ so we can apply Master Method to solve this recursion.
As

$$n^{\log_b a} = n^{\log_{\frac{3}{2}} 1} = n^0 = 1$$

is polynomially equal to $f(n) = 1$.
so $n^0 = \Theta\left(n^{\log_{\frac{3}{2}} 1}\right) = \Theta(n^0) = \Theta(1)$.
Case 2 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(n^{\log_{\frac{3}{2}} 1} \lg n\right) = \Theta\left(n^0 \lg n\right) = \Theta(\lg n)$$

3. Find the solution of $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$

Comparing $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ $a = 3 \geq 1, b = 4 > 1$ and $f(n) = n\lg n$ so we can apply Master Method to solve this recursion.
As $n^{\log_b a} = n^{\log_4 3} = n^{0.79}$ is polynomially less than $f(n) = n\lg n$.
so $n\lg n = \Omega(n^{(\log_4 3+\epsilon)}) = n^{(0.79+\epsilon)}$ for $\epsilon = 0.2$
Also

$$3f(n/4) = 3\frac{n}{4}\lg\left(\frac{n}{4}\right)$$

$$\Rightarrow \frac{3}{4}n\lg\left(\frac{n}{4}\right)$$

$$\Rightarrow \frac{3}{4}n\lg n - \frac{3}{4}n\lg 4$$

Solution of Examples 3 of Master method (D&C) II

$$\Rightarrow \frac{3}{4} n \lg n - \frac{3}{2} n$$

$$\Rightarrow \frac{3}{4} n \lg n - \frac{3}{2} n \leq cn \lg n$$

where $\frac{3}{4} \leq c < 1$.
Case 3 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(f(n)\right) = \Theta\left(n \lg n\right)$$

Solution of Examples 4 of Master method (D&C)

4. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$

Comparing $T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 2 \geq 1, b = 2 > 1$ and $f(n) = n \lg n$ so we can apply Master Method to solve this recursion.
As $n^{\log_b a} = n^{\log_2 2} = n^1$ is not polynomially less than, greater than or equal to $f(n) = n \lg n$. It is asymptotically less than $n \lg n$. It lies between the gaps of Case 2 and Case 3.
So it cann't be solved using Master Method.

5. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$

Comparing $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 2 \geq 1, b = 2 > 1$ and $f(n) = \frac{n}{\lg n}$ so we can apply Master Method to solve this recursion.
As $n^{\log_b a} = n^{\log_2 2} = n^1$ is not polynomially less than, greater than or equal to $f(n) = \frac{n}{\lg n}$. It is asymptotically greater than $\frac{n}{\lg n}$. It lies between the gaps of Case 1 and Case 2.
So it cann't be solved using Master Method.

Solution of Examples 6 of Master method (D&C)

6. Find the solution of $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Comparing $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 2 \geq 1, b = 2 > 1$ and $f(n) = \Theta(n) = cn$ so we can apply Master Method to solve this recursion.
As

$$n^{\log_b a} = n^{\log_2 2} = n^1$$

is polynomially equal to $f(n) = cn$.
so $cn = \Theta(n)$.
Case 2 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(n^1 \lg n\right) = \Theta(n \lg n)$$

Solution of Examples 7 of Master method (D&C)

7. Find the solution of $T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$

Comparing $T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 8 \geq 1, b = 2 > 1$ and $f(n) = \Theta(n^2) = cn^2$ so we can apply Master Method to solve this recursion.
As

$$n^{\log_b a} = n^{\log_2 8} = n^3$$

is polynomially greater than $f(n) = cn^2$.
so $cn^2 = O(n^{3-\epsilon})$ for $0 < \epsilon \leq 1$
Case 1 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(n^3\right)$$

8. Find the solution of $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^3)$

Comparing $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^3)$ with $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
$a = 4 \geq 1, b = 2 > 1$ and $f(n) = \Theta(n^3) = dn^3, d > 0$ so we can apply Master Method to solve this recursion.
As

$$n^{\log_2 4} = n^2$$

is polynomially less than $f(n) = dn^3$.
Also $af(\frac{n}{b}) \leq cf(n)$ for $0 < c < 1$

$$\Rightarrow 4d\left(\frac{n}{2}\right)^3$$

$$\Rightarrow \frac{4}{8}dn^3$$

Solution of Examples 8 of Master method (D&C) II

$$\Rightarrow \frac{1}{2}dn^3 \leq cdn^3$$

Where $\frac{1}{2} \leq c < 1, \Rightarrow c = 0.75$
so $dn^3 = \Omega(n^{2+\epsilon})$ for $0 < \epsilon \leq 1$
Case 3 of Master Method will be applied and solution will be

$$T(n) = \Theta\left(n^3\right)$$

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598