Computability

Module 5

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Outline

Partial and Total Functions

- A Partial Function f from X to Y $(f : X \to Y)$ is a rule which assigns to every element of X at most one element of Y.
- Example: if R denotes the set of all real numbers, the rule f from R to R given by $f(r) = +\sqrt{r}$; is a partial function since $f(r)$ is not defined as a real number when r is negative.
- A Total Function f from X to Y is a rule which assigns to every element of X a unique element of Y.
- Example:The rule f from R to R given by $f(r) = |r|$ is a total function since $f(r)$ is defined for every real number r.
- We consider total functions f from X^k to X, where $X = \{0, 1, 2, 3, \dots\}$ or $X = \{a, b\}^*$.
- We denote $\{0, 1, 2, \dots\}$ by N and $\{a, b\}$ by Σ.
- X^k is the set of all k-tuples of elements of X.

Partial and Total Functions

- For example, $f(m,n) = m - n$ defines a partial function from N to itself as $f(m,n)$ is not defined when $m - n < 0$.
- But $g(m,n) = m + n$ defines a total function from N to itself.
- A partial or total function f from X^k to X is also called a function of k variables and denoted by $f(x_1, X_2, \ldots, X_k)$.
- For example, $f(x_1, x_2) = 2x_1 + x_2$ is a function of two variables: $f(1,2) = 4; 1$ and 2 are called arguments and 4 is called a value.
- $g(w_1, w_2) = w_1 w_2$ is a function of two variables $w_1, w_2 \in \Sigma^*$: $g(ab, aa) = abaa, ab, aa$ are called arguments and $abaa$ is a value.

Primitive Recursive functions

- The initial functions over N are given as:
  1. Zero function Z defined by $Z(x) = 0$
  2. Successor function S defined by $S(x) = x + 1$
  3. Projection function U_i^n defined by $U_i^n(x_1, \ldots, x_n) = x_i$
  4. As $U_1^1(x) = x$ for every x in N. U_1^1 is simply the identity function. So U_i^n is also termed a generalized identity function.
- The initial functions over Σ are given as:
  1. nil(x) defined by $nil(x) = \wedge$
  2. cons a(x) defined by $cons\ a(x) = ax$
  3. cons b(x) defined by $cons\ b(x) = bx$

Primitive Recursive functions

- Example:

$$Z(7) = 0$$

$$S(4) = 5$$

$$U_2^3\{2, 5, 7\} = 5$$

$$nil(aabb) = \wedge$$

$$cons\ a(aabb) = aaabb$$

$$cons\ b(aabb) = baabb$$

Primitive Recursive functions

- Composition of a function: If $f_1, f_2, \ldots, f_k$ are partial functions of n variables and g is a partial function of k variables, then the composition of g with $f_1, f_2, \ldots, f_k$ is a partial function of n variables defined by

$$g(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_k(x_1, x_2, \ldots, x_n))$$

- The composition of g with $f_1, f_2, \ldots, f_n$ is total when $g, f_1, f_2, \ldots, f_n$ are total.
- Example: Let $f_1(x, y) = x + y, f_2(x, y) = 2x, f_3(x, y) = xy$ and $g(x, y, z) = x + y + z$ be functions over N. Find the composition of g with f_1, f_2, f_3
- Solution: The composition of g with f_1, f_2, f_3 is given by $h(x, y) = g(f_1(x, y), f_2(x, y), f_3(x, y)) = (x + y) + (2x) + (xy)$ $= x + y + 2x + xy$

Primitive Recursive functions

- A function $f(x)$ over N is defined by recursion if there exists a constant k (a natural number) and a function $h(x, y)$ such that $f(0) = k, f(n+1) = h(n, f(n)$
- Example: Define $n!$ by recursion.
- Solution: Let $f(0) = 1$ and $f(n+1) = h(n, f(n))$, where $h(x, y) = S(x) * y$.
  So $f(n)$ will be
  $f(n) = h(n-1, f(n-1)) = S(n-1) * f(n-1) = n * f(n-1)$
- A function f of $n+1$ variables is defined by recursion if there exists a function g of n variables, and a function h of $n+2$ variables, and f is defined as follows:
  $f(x_1, x_2, \ldots, x_n, 0) = g(x_1, x_2, \ldots, x_n)$
  $f(x_1, x_2, \ldots, x_n, y+1) =$
  $h(x_1, x_2, \ldots, x_n, y, f(x_1, x_2, \ldots, x_n, y))$

Primitive Recursive functions

- A total function f over N is called primitive recursive
  (i) if it is anyone of the three initial functions, or
  (ii) if it can be obtained by applying composition and recursion a finite number of times to the set of initial functions.
- A total function is primitive recursive if it can be obtained by applying composition and recursion a finite number of times to primitive recursive functions $f_1, f_2, \ldots, f_m$. Each f_i is obtained by applying composition and recursion a finite number of times to initial functions.

Primitive Recursive functions

- A function $f(x)$ over Σ is defined by recursion if there exists a 'constant' string $w \in \Sigma^*$ and functions $h_1(x, y)$ and $h_2(x.y)$ such that

$$f(\wedge) = w$$

$$f(ax) = h_1(x, f(x))$$

$$f(bx) = h_2(x, f(x))$$

h_1 and h_2 may be functions in one variable.

Primitive Recursive functions

- A function $f(x_1, x_2, \ldots, x_n)$ over Σ is defined by recursion if there exists a function $g(x_1, x_2, \ldots, x_{n-1})$, $h_1(x_1, x_2, \ldots, x_{n+1})$, $h_2(x_1, x_2, \ldots, x_{n+1})$ such that

$$f(\wedge, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$$

$$f(ax_1, x_2, \ldots, x_n) = h_1(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

$$f(bx_1, x_2, \ldots, x_n) = h_2(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

  h_1 and h_2 may be functions of m variables, where $m < n + 1$.

- A total function f over Σ is primitive recursive
  (i) if it is anyone of the three initial functions, or
  (ii) if it can be obtained by applying composition and recursion a finite number of times to the initial functions.

Recursive functions

- Let $g(x_1, x_2, \ldots, x_n, y)$ be a total function over N. The function g is a regular function if there exists some natural number y_0 such that $g(x_1, x_2, \ldots, x_n, y_0) = 0$ for all values $x_1, x_2, \ldots, x_n \in N$.
- Example: $g(x, y) = min(x, y)$ is a regular function since $g(x, 0) = 0$ for all $x \in N$.
- But $f(x, y) = |x - y|$ is not regular since $f(x, y) = 0$ only when $x = y$, and so we cannot find a fixed y such that $f(x, y) = 0$ for all x in N.

Recursive functions

- A function $f(x_1, x_2, \ldots, x_n)$ over N is defined from a total function $g(x_1, x_2, \ldots, x_n, y)$ by minimization if
  (a) $f(x_1, x_2, \ldots, x_n)$ is the least value of all y's such that $g(x_1, x_2, \ldots, x_n, y) = 0$ if it exists. The least value is denoted by $\mu_y(g(x_1, x_2, \ldots, x_n, y) = 0)$.
  (b) $f(x_1, x_2, \ldots, x_n)$ is undefined if there is no y such that $g(x_1, x_2, \ldots, x_n, y) = 0$
- In general, f is partial. But, if g is regular then f is total.
- A function is recursive if it can be obtained from the initial functions by a finite number of applications of composition, recursion and minimization over regular functions.
- A function is partial recursive if it can be obtained from the initial functions by a finite number of applications of composition, recursion and minimization.

Recursive functions

- Example: Show that $f(x) = x/2$ is a partial recursive function over N.
- Solution: Let $g(x, y) = |2y - x|$ where $2y - x = 0$ for some y only when x is even. Let $f_1(x) = \mu_y(|2y - x| = 0)$. Then $f_1(x)$ is defined only for even values of x and is equal to $x/2$. When x is odd, $f_1(x)$ is not defined $f_1(x)$ is partial recursive. As $f(x) = x/2 = f_1(x)$ is a partial recursive function.
- Exercise: Show that $f(x, y) = x^2y^4 + 7xy^3 + 4y^5$ is primitive recursive.

Questions?

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Thank you.