Algorithms Design and Analysis [ETCS-301]

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September 17, 2019

Floyd Warshall algorithm

- The algorithm find the shortest route between all nodes/vertices of the non-negative directed weighted graph G.
- Suppose V is set of n nodes/vertices in graph G i.e. $\{1, 2, \ldots, n\} \in V$
- E is the set of edges of the graph G .i.e $(i, j) \in E$ if there is a edge from i to j in the graph
- So we can say $G = (V, E)$
- w_{ij} is the weight of the edge from vertex i to j.
- if p is a path from vertex v_i to v_j through vertices $v_1, v_2, \ldots, v_l$ then the nodes other than source and destination is known as intermediate nodes.
- Sum of weight of the edges of the path p is the distance d_{ij} between vertex v_i to v_j
- if this distance is the shortest among all possible paths from v_i to v_j then it is called the shortest path $\delta(i, j)$

Step 1: The structure of a shortest path I

- Let graph G has set of vertices $V = \{1, 2, \ldots, n\}$
- Let any pair of vertices $i, j \in V$, all paths can be drawn from the set of intermediate vertices $\{1, 2, \ldots, k\}$ for some k
- Let p is the shortest/minimum-weight simple path from i to j.
- There are two cases for any vertex k : either k is the part of the minimum-weight path or not the part of the path.
- If k is not the part of the path then it can be removed from the the set of vertices without effecting the path. So the path will be from $\{1, 2, \ldots, k-1\}$ or we can say a shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \ldots, k-1\}$ is also a shortest path from i to j with all intermediate vertices in the set $\{1, 2, \ldots, k\}$
- But if it is part of the path then we can divide the whole path p into two paths: p_1 - from i to k and p_2 - from k to j

Step 1: The structure of a shortest path II

- Intermediate nodes of p_1 and p_2 will be from $\{1, \ldots, k-1\}$ because now k is a source in one path and destination in other but not the intermediate node.

$$\delta(i,j) = \begin{cases} \delta(i,j) & \text{if } k \text{ is not part of } p \\ \delta(i,k) + \delta(k,j) & \text{if } k \text{ is part of } p \end{cases}$$

with intermediate nodes $\{1, 2, \ldots, k-1\}$

Step 2: A recursive solution I

- d_{ij}^{k} is the shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \ldots, k\}$
- if $k = 0$ then no intermediate node so $d_{ij}^{0} = w_{ij}$ if there is a direct edge from i to j else $d_{ij}^{0} = \infty$ if there is no direct edge from i to j and $d_{ij}^{0} = 0$ if $i = j$.
- $d_{ij}^{k} = \min\{d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}\}$
- π_{ij}^{k} is the predecessor of vertex j for the shortest path from vertex i with all intermediate vertices in the set $\{1, 2, \ldots, k\}$
- if $k = 0$ then no intermediate node so $\pi_{ij}^{0} = i$ if there is a direct edge from i to j else $\pi_{ij}^{0} = NIL$ if there is no direct edge from i to j and $\pi_{ij}^{0} = NIL$ if $i = j$.

Step 2: A recursive solution II

- $\pi_{ij}^{k} = \pi_{ij}^{k-1}$ if $d_{ij}^{k-1} \leq d_{ik}^{k-1} + d_{kj}^{k-1}$ or $\pi_{ij}^{k} = \pi_{kj}^{k-1}$ if $d_{ij}^{k-1} > d_{ik}^{k-1} + d_{kj}^{k-1}$

$$d_{ij}^{k} = \begin{cases} 0 & \text{if } i = j, k = 0 \\ \infty & \text{if } (i,j) \notin E, k = 0 \\ w_{ij} & \text{if } (i,j) \in E, k = 0 \\ \min\{d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}\} & \end{cases}$$

$$\pi_{ij}^{k} = \begin{cases} NIL & \text{if } i = j, k = 0 \\ NIL & \text{if } w_{ij} = \infty, k = 0 \\ i & \text{if } w_{ij} < \infty, k = 0 \\ \pi_{ij}^{k-1} & \text{if } d_{ij}^{k-1} \leq d_{ik}^{k-1} + d_{kj}^{k-1} \\ \pi_{kj}^{k-1} & \text{if } d_{ij}^{k-1} > d_{ik}^{k-1} + d_{kj}^{k-1} \end{cases}$$

Let W is the weight matrix and A is the adjacency matrix of the graph G of n vertices.

Bottom-up Approach-I

FLOYD-WARSHALL(W,A,n)

1. $D^0 = W$
2. $\Pi^0 = A$
3. for $k = 1$ to n
4. Let $D^k = (d_{ij}^k)$ a $n \times n$ matrix
5. Let $\Pi^k = (\pi_{ij}^k)$ a $n \times n$ matrix
6. for $i = 1$ to n
7. for $j = 1$ to n
8. if $d_{ij}^{k-1} \leq d_{ik}^{k-1} + d_{kj}^{k-1}$
9. $d_{ij}^k = d_{ij}^{k-1}$

Step 3: Computation of Shortest path II

$\pi_{ij}^{k} = \pi_{ij}^{k-1}$

else

$d_{ij}^{k} = d_{ik}^{k-1} + d_{kj}^{k-1}$

$\pi_{ij}^{k} = \pi_{kj}^{k-1}$

return D^n, Π^n

Step 3: Computation of Shortest path III

Bottom-up Approach-II

FLOYD-WARSHALL(W,A,n)

$D = W$
$\Pi = A$
for $k = 1$ to n
    for $i = 1$ to n
        for $j = 1$ to n
            if $d_{ij} \leq d_{ik} + d_{kj}$
                $d_{ij} = d_{ij}$
                $\pi_{ij} = \pi_{ij}$
            else
                $d_{ij} = d_{ik} + d_{kj}$
                $\pi_{ij} = \pi_{kj}$
return D, Π

Thank you

Please send your feedback or any queries to akyadav1@amity.edu, akyadav@akyadav.in or contact me on +91 9911375598